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Quantum and Classical Brackets

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We describe p-mechanical (Kisil, V. V. (1996). *Journal of Natural Geometry* **9**(1), 1–14; Kisil, V. V. (1999). *Advances in Mathematics* **147**(1), 35–73; Prezhdo, O. V. and Kisil, V. V. (1997). *Physical Review A* **56**(1), 162–175) brackets that generate quantum (commutator) and classical (Poisson) brackets in corresponding representations of the Heisenberg group. We *do not* use any kind of semiclassical approximation or limiting procedure for $\hbar \to 0$.

1. INTRODUCTION

The purpose of this short announcement is to describe "brackets" in a p-mechanical setting (Kisil, 1996, 1999; Prezhdo and Kisil, 1997), which generates both classical (Poisson) and quantum (commutator) brackets. Consequently we are able to derive dynamical equation in classical and quantum cases from the same consistent source.

The principal step in transition from Lagrangian to Hamiltonian mechanics is the introduction by means of the Legendre transform of *new independent variables*—coordinates and momenta—instead of coordinates and depending from them their time derivatives—the velocities $\dot{q}$. Similarly the p-mechanical construction (Kisil, 1996, 1999; Prezhdo and Kisil, 1997) is based on the introduction by means of the Fourier transform of new variables (s, x, y) such that (x, y) is Fourier dual to (q, p) and s is Fourier dual to the Planck constant $\hbar$. It appeared that points (s, x, y) are elements of the Heisenberg group $\mathbb{H}^n$ (Howe, 1980a,b; Taylor, 1986) (see also (2.5)).

It is known from the works of von Neumann that the Heisenberg picture of quantum mechanics is generated by infinite dimensional noncommutative irreducible unitary representations of $\mathbb{H}^n$ (Howe, 1980b). But one-dimensional (commutative!) unitary representations of $\mathbb{H}^n$ are often not employed. It is shown within p-mechanical framework that these one-dimensional representations

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contain classical dynamics exactly in the same way as infinite-dimensional ones—quantum.

An important feature of our approach is that we *do not* use any kind of semi-classical approximation or limiting procedure for $\hbar \to 0$. The classical picture is not any more an imperfect shade of the "correct" quantum description—both quantum and classical pictures stand on the equal ground.

Here we present a p-mechanical version of brackets and a dynamical equation generated by them. Our considerations are illustrated by a simple example of harmonic oscillator. More involved examples allowing mix quantum and classical components within one system will be presented elsewhere.

2. PRELIMINARIES

2.1. Groups and Their Representations

We consider $L_2(\mathbb{R}^n)$ equipped with the scalar product

$$\langle f, g\rangle = \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} f(x)\bar{g}(x)\, dx. \tag{2.1}$$

Throughout the paper we use the standard notation for the Fourier transform:

$$[\mathcal{F} f](\hbar) = \hat{f}(\hbar) = \sqrt{2\pi} \int_{-\infty}^{\infty} f(s)\, e^{-is\hbar}\, ds.$$

Let G be a group with an invariant measure dg. $L_1(G, dg)$ could be upgraded from a linear space to an algebra with the convolution multiplication:

$$(k_1 * k_2)(g) = \int_G k_1(h)\, k_2(h^{-1}g)\, dh = \int_G k_1(gh^{-1})\, k_2(h)\, dh. \tag{2.2}$$

Let ρ be a representation of G (Taylor, 1986, Chap. 1), we will work mainly with unitary irreducible ones. We could extend ρ to $L_1(G, dg)$ by the formula:

$$\rho(k) = \int_G k(g)\rho(g)\, dg. \tag{2.3}$$

From the general properties of representations of Lie groups (Taylor, 1986, Chap. 1, (2.17)) we have

$$\rho(k_1) + \lambda\rho(k_2) = \rho(k_1 + \lambda k_2), \qquad \rho(k_1)\rho(k_2) = \rho(k_1 * k_2). \tag{2.4}$$

This could be reinforced in the following statement.

Lemma 2.1. (*Algebraic Inheritance*). *Let $p(a_1, a_2, \ldots, a_n)$ be a polynomial in non-commuting arguments $a_1, a_2, \ldots, a_n$. Let functions $k_1, k_2, \ldots, k_n$ from $L_1(G)$*

satisfy to the identity

$$p(k_1, k_2, \ldots, k_n) = 0,$$

where multiplication is defined as the group convolution on G. Then

$$p(\rho(k_1), \rho(k_2), \ldots, \rho(k_n)) = 0$$

for an arbitrary representation ρ of G.

2.2. The Heisenberg Group $\mathbb{H}^n$ and its Representations

Let (s, x, y), where $x, y \in \mathbb{R}^n$ and $s \in \mathbb{R}$, be an element of the Heisenberg group $\mathbb{H}^n$ (Howe, 1980a,b; Taylor, 1986). The group law on $\mathbb{H}^n$ is given as follows:

$$(s, x, y) * (s', x', y') = \left(s + s' + \frac{1}{2}(xy' - x'y), x + x', y + y'\right). \tag{2.5}$$

For our purpose we need all irreducible representations of the group $\mathbb{H}^n$. They are given by the following famous theorem:

Theorem 2.2. (*Stone–von Neumann*) (*Kirillov, 1976, §18.4; Taylor, 1986, § 1.2*). *All unitary irreducible representations of the Heisenberg group $\mathbb{H}^n$ up to unitary equivalence are as follows:*

(i) *For any $\hbar \in (0, \infty)$ the Schrödinger irreducible noncommutative unitary representations in $L_2(\mathbb{R}^n)$*

$$\rho_{\pm\hbar}(s, x, y) = e^{i(\pm s\cdot\hbar I \pm x\cdot\hbar^{1/2}M + y\cdot\hbar^{1/2}D)}, \tag{2.6}$$

where xM and yD are such unbounded self-adjoint operators on $L_2(\mathbb{R}^n)$:

$$(x \cdot \hbar^{1/2}M)\, u(\upsilon) = \hbar^{1/2} \sum x_j \upsilon_j\, u(\upsilon), \tag{2.7}$$

$$(y \cdot \hbar^{1/2}D)\, u(\upsilon) = \frac{\hbar^{1/2}}{i} \sum y_j \frac{\partial u}{\partial \upsilon_j}. \tag{2.8}$$

Representation (2.6) *acts on a function $u(\upsilon)$ as follows:*

$$\rho_{\pm\hbar}(s, x, y)\, u(\upsilon) = e^{i(\pm(s+xy/2)\cdot\hbar I \pm x\cdot\hbar^{1/2}\upsilon)}\, u\big(\upsilon + \hbar^{1/2}y\big). \tag{2.9}$$

(ii) *For $(q, p) \in \mathbb{R}^{2n}$ commutative one-dimensional representations on $\mathbb{C}$:*

$$\rho_{(q,p)}(s, x, y)\, u = e^{i(qx+py)}u, \quad u \in \mathbb{C}. \tag{2.10}$$

In some sense (Kisil, 1996) the last representations (2.10) correspond to the case $\hbar = 0$. While other representations of $\mathbb{H}^n$ could be transformed to the above ones by unitary operators it is better sometime to stay with alternative forms tailored to particular models. For example, the Segal–Bargmann representation (Bargmann, 1961; Segal, 1963) is well suited for quantum field theory and its relation to the Schrödinger representation (2.6) illuminate many results in analysis and quantum theory (Howe, 1980b).

Representations (2.6)–(2.10) generate accordingly to (2.3) representations of convolution algebra $L_1(\mathbb{H}^n)$ expressed by formulas (Taylor, 1986, Chap. 1, (3.9)):

$$\rho_{\pm\hbar}[k(s,x,y)] = \hat{k}(\pm\hbar, \pm\hbar^{1/2}M, \hbar^{1/2}D), \tag{2.11}$$

$$\rho_{(q,p)}[k(s,x,y)] = \hat{k}(0,q,p). \tag{2.12}$$

The right side of (2.11) specifies a pseudodifferential operator (PDO) (Hörmander, 1985; Shubin, 1987) with the Weyl symbol $\hat{k}(\pm\hbar, \pm\hbar^{1/2}x, \hbar^{1/2}\xi)$. Such a PDO with a symbol $a(\upsilon, \nu)$ is defined by

$$a_\tau(M,D)\,u(\upsilon) = (2\pi)^{-N}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} e^{i\langle \upsilon-u,\nu\rangle}\, a(\tau u + (1-\tau)\upsilon, \nu)\, u(u)\, d\nu\, du. \tag{2.13}$$

The right side of (2.12) is just a constant from $\mathbb{C}$.

Using (2.4) with ρ equal either to $\rho_\hbar$ (2.6) or to $\rho_{(q,p)}$ (2.10) we obtain

$$\rho(k_1 * k_2 - k_2 * k_1) = \begin{cases} [K_1, K_2] = K_1K_2 - K_2K_1, & \rho = \rho_\hbar, \hbar \neq 0; \\ 0, & \rho = \rho_{(q,p)}, \end{cases} \tag{2.14}$$

where operators K_1 and K_2 are Weyl PDO defined by (2.11) for functions k_1 and k_2, respectively.

3. QUANTUM AND CLASSICAL BRACKETS

3.1. *p*-Mechanical Brackets and its Quantum and Classical Representations

Let $L_1^\upsilon(\mathbb{R})$ be the linear subspace of L_1 functions on $\mathbb{R}$ such that

$$\lim_{s\to-\infty} s\int_{-\infty}^{s} f(t)\,dt = 0, \quad \text{and} \quad \lim_{s\to\infty} s\int_{s}^{\infty} f(t)\,dt = 0.$$

A nontrivial function from $L_1^\upsilon(\mathbb{R})$ is, for example, xe^{-x^2}. The following could be easily seen (cf. (Kirillov and Gvishiani, 1982, § IV.1.1, and § IV.2.3)).

Lemma 3.1. (i) $L_1^\upsilon(\mathbb{R})$ *is a* closed ideal *in convolution algebra* $L_1(\mathbb{R})$.

(ii) *The Fourier transform of functions from* $L_1^\upsilon(\mathbb{R})$ *are among continuous functions such that* $\hat{f}(0) = 0$.

Let $\mathcal{A}$ be an anti-derivation—linear unbounded operator from $L_1^{\upsilon}(\mathbb{R})$ onto the space of integrable functions on $\mathbb{R}$ defined by the formula:

$$[\mathcal{A} f](s) = \int_{-\infty}^{s} f(t)\, dt = \int_{-\infty}^{\infty} \chi(s-t)\, f(t)\, dt, \tag{3.1}$$

where $\chi(t)$ is the Heaviside function:

$$\chi(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ 1, & \text{if } t > 0. \end{cases} \tag{3.2}$$

From the definition it follows that

Lemma 3.2. *The antiderivative $\mathcal{A}$ enjoys the following properties:*

(i) $\mathcal{A}\mathbf{0} = \mathbf{0}$, *where* $\mathbf{0}$ *is the function identically equal to* 0*. The function* 0 *is the only element of the kernel of* $\mathcal{A}$*:* $\ker \mathcal{A} = \{\mathbf{0}\}$*;*

(ii) $\mathcal{A}$ *commutes with all shifts* $f(s) \to f(s+a)$ *and their linear combinations—convolution operators on* $\mathbb{R}$*.*

(iii) *For* $f \in L_1^{\upsilon}(\mathbb{R})$ *the limits at infinity vanish:*

$$\begin{aligned} \lim_{s\to-\infty} [\mathcal{A} f](s) = \lim_{s\to\infty} [\mathcal{A} f](s) &= \lim_{s\to-\infty} s[\mathcal{A} f](s) \\ &= \lim_{s\to\infty} s[\mathcal{A} f](s) = 0. \end{aligned} \tag{3.3}$$

(iv) *If* $f_1, f_2 \in L_1^{\upsilon}(\mathbb{R})$ *then* $\mathcal{A}(f_1 * f_2) = (\mathcal{A} f_1) * f_2 = f_1 * (\mathcal{A} f_2)$ *is again in* $L_1^{\upsilon}(\mathbb{R})$*.*

From integration by parts:

$$\int_{-\infty}^{\infty} [\mathcal{A} f](s)\, e^{-is\hbar}\, ds = [\mathcal{A} f](s) \frac{e^{-is\hbar}}{-i\hbar} \bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(s) \frac{e^{-is\hbar}}{-i\hbar}\, ds$$

and (3.3) we obtain

$$\mathcal{F}[\mathcal{A} f](\hbar) = \begin{cases} \frac{1}{i\hbar}[\mathcal{F} f](\hbar), & \hbar \neq 0; \\ -\sqrt{2\pi} \int_{-\infty}^{\infty} f(s)\, s\, ds, & \hbar = 0, \end{cases} \tag{3.4}$$

for $f(s) \in L_1^{\upsilon}(\mathbb{R})$. In fact we could take the last formulae as a definition of the operator $\mathcal{A}$.

Definition 3.3. The *p-mechanical brackets* of two functions $k_1(s, x, y)$, $k_2(s, x, y)$ on the Heisenberg $\mathbb{H}^n$ are defined as follows:

$$\{\![k_1, k_2]\!\} = \mathcal{A}(k_1 * k_2 - k_2 * k_1), \tag{3.5}$$

where $*$ denotes the group convolution on $\mathbb{H}^n$ of two functions and $\boldsymbol{\mathcal{A}}$ acts as an antiderivative with respect of the variable s.

This definition of the p-mechanical bracket has sense if $k_{1,2}(s, x_0, y_0) \in L_1^v(\mathbb{R})$ for any fixed $x_0, y_0 \in \mathbb{R}^n$. Because of Lemma 3.2.(iv) the p-brackets of two such functions is again in $L_1^v(\mathbb{R})$, thus $\boldsymbol{\mathcal{A}}$ is meaningful in (3.5). While this completely serves the purpose of the present paper future extensions of Definition 3.3 are also possible. Note also, that we put L_1^v-condition only with respect to variable s; variables x and y, which are Fourier-dual to physical coordinates and momenta, are unrestricted.

Lemma 3.4. *The p-mechanical brackets* (3.5) *have the following properties:*

(i) *They are linear.*
(ii) *They are antisymmetric* $\{\![k_1, k_2]\!\} = -\{\![k_2, k_1]\!\}$.
(iii) *They satisfy to the Jacoby identity*

$$\{\![\{\![k_1, k_2]\!\}, k_3]\!\} + \{\![\{\![k_2, k_3]\!\}, k_1]\!\} + \{\![\{\![k_3, k_1]\!\}, k_2]\!\} = 0. \tag{3.6}$$

(iv) *They are a derivation, i.e. satisfy to the Leibniz rule:*

$$\{\![k_1 * k_2, k_3]\!\} = \{\![k_1, k_3]\!\} * k_2 + k_1 * \{\![k_2, k_3]\!\}. \tag{3.7}$$

Proof: The linearity and antisymmetric properties are obvious. Two other properties are secured because

(i) $\boldsymbol{\mathcal{A}}$ commutes with convolutions (Lemma 3.2. (ii)) and sends zero function to itself (Lemma 3.2.(i));
(ii) The commutator $k_1 * k_2 - k_2 * k_1$ satisfies both to Jacoby and Leibniz identity.

For example the Leibniz identity could be verified as follows:

$$\begin{aligned}
\{\![k_1 * k_2, k_3]\!\} &= \boldsymbol{\mathcal{A}}(k_1 * k_2 * k_3 - k_3 * k_1 * k_2)\\
&= \boldsymbol{\mathcal{A}}(k_1 * k_2 * k_3 - k_1 * k_3 * k_2 + k_1 * k_3 * k_2 - k_3 * k_1 * k_2)\\
&= \boldsymbol{\mathcal{A}}(k_1 * k_2 * k_3 - k_1 * k_3 * k_2)\\
&\quad + \boldsymbol{\mathcal{A}}(k_1 * k_3 * k_2 - k_3 * k_1 * k_2) \qquad (3.8)\\
&= k_1 * \boldsymbol{\mathcal{A}}(k_2 * k_3 - k_3 * k_2) + \boldsymbol{\mathcal{A}}(k_1 * k_3 - k_3 * k_1) * k_2 \qquad (3.9)\\
&= k_1 * \{\![k_2, k_3]\!\} + \{\![k_1, k_3]\!\} * k_2,
\end{aligned}$$

where (3.8) follows from the linearity of $\boldsymbol{\mathcal{A}}$ and (3.9) is a consequence of Lemma 3.2.(ii). □

Now we describe image of the brackets under representations of $\mathbb{H}^n$.

Proposition 3.5. *The images of p-mechanical brackets* (3.5) *under infinite dimensional representations* $\rho_\hbar$, $\hbar \neq 0$ *and finite dimensional representations* $\rho_{(q,p)}$ *are quantum commutant and Poisson brackets of functions* $\hat{k}_1$ *and* $\hat{k}_2$*, respectively:*

$$\rho(\{[k_1, k_2]\}) = \begin{cases} \frac{1}{i\hbar}[\hat{k}_1, \hat{k}_2] = \frac{1}{i\hbar}(K_1 K_2 - K_2 K_1), & \rho = \rho_\hbar, \quad \hbar \neq 0; \\ \{\hat{k}_1, \hat{k}_2\} = \frac{\partial \hat{k}_1}{\partial q}\frac{\partial \hat{k}_2}{\partial p} - \frac{\partial \hat{k}_1}{\partial p}\frac{\partial \hat{k}_2}{\partial q}, & \rho = \rho_{(q,p)}. \end{cases} \tag{3.10}$$

Proof: The proof is a straightforward calculation using (3.4). We will carry them separately for cases of $\hbar \neq 0$ and $\hbar = 0$.

Let $\rho = \rho_\hbar$, $\hbar \neq 0$. Then

$$\begin{aligned} \rho_\hbar(\{[k_1, k_2]\}) &= \int_{\mathbb{H}_n} \{[k_1, k_2]\}(g)\, \rho_\hbar(g)\, dg \\ &= \int_{\mathbb{H}^n} \boldsymbol{\mathcal{A}}(k_1 * k_2 - k_2 * k_1)\,(s, x, y)\, e^{i(\pm \hbar s I \pm \hbar^{1/2} x M + \hbar^{1/2} y D)}\, dg \\ &= \frac{1}{i\hbar}\int_{\mathbb{H}^n} (k_1 * k_2 - k_2 * k_1)\,(s, x, y)\, e^{i(\pm \hbar s I \pm \hbar^{1/2} x M + \hbar^{1/2} y D)}\, dg \end{aligned} \tag{3.11}$$

$$= \frac{1}{i\hbar}[K_1, K_2], \tag{3.12}$$

where the line (3.11) follows from the first case in (3.4) and (3.12) is exactly the first case in (2.14).

The second case $\rho = \rho_{(q,p)}$ (symbolically corresponding to "$\hbar = 0$") is also not difficult but somehow longer:

$$\begin{aligned} \rho_{(q,p)}(\{[k_1, k_2]\}) &= \int_{\mathbb{H}^n} \{[k_1, k_2]\}(g)\, \rho_{(q,p)}(g)\, dg \\ &= \int_{\mathbb{H}^n} \boldsymbol{\mathcal{A}}(k_1 * k_2 - k_2 * k_1)\,(s, x, y)\, e^{i(qx+py)}\, dg \\ &= \int_{\mathbb{H}^n} (k_2 * k_1 - k_1 * k_2)\,(s, x, y)\, s e^{i(qx+py)}\, dg \qquad (3.13) \\ &= \int_{\mathbb{H}^n}\int_{\mathbb{H}^n} \left(k_2(s', x', y')\, k_1\left(s - s' + \frac{x'y - xy'}{2},\right.\right. \\ &\qquad \left. x - x', y - y'\right) - k_1(s', x', y')\, k_2\left(s - s' + \frac{x'y - xy'}{2},\right. \\ &\qquad \left.\left. x - x', y - y'\right)\right) dg' s e^{i(qx+py)}\, dg. \end{aligned}$$

We use the second case of (3.4) to obtain (3.13). Now let us change variables

$$\begin{aligned} x'' &= x - x', \qquad y'' = y - y', \qquad s'' = s - s' + \frac{x'y - xy'}{2}; \\ x' &= x'' + x', \qquad y = y'' + y', \qquad s = s'' + s' + \frac{x''y' - x'y''}{2}, \end{aligned} \tag{3.14}$$

and continue the above calculations:

$$= \int_{\mathbb{H}^n} \int_{\mathbb{H}^n} (k_2(s', x', y')\, k_1(s'', x'', y'') - k_1(s', x', y')\, k_2(s'', x'', y''))$$
$$\times \left(s'' + s' + \frac{x''y' - x'y''}{2} \right) e^{i(q(x''+x')+p(y''+y'))}\, dg'\, dg''$$
$$= \int_{\mathbb{H}^n} \int_{\mathbb{H}^n} (k_2(s', x', y')\, k_1(s'', x'', y'') - k_1(s', x', y')\, k_2(s'', x'', y'')) \tag{3.15}$$
$$\times (s'' + s')\, e^{i(qx'+py')}\, e^{i(qx''+py'')}\, dg'\, dg'' \tag{3.16}$$
$$+ \int_{\mathbb{H}^n} \int_{\mathbb{H}^n} (k_2(s', x', y')\, k_1(s'', x'', y'') - k_1(s', x', y')\, k_2(s'', x'', y'')) \tag{3.17}$$
$$\times \frac{x''y' - x'y''}{2}\, e^{i(qx'+py')}\, e^{i(qx''+py'')}\, dg'\, dg''. \tag{3.18}$$

Interchanging primed and double primed variables in (3.15) and (3.16) we conclude that the integral is equal to itself with the opposite sign and thus vanish. In contrast such an interchange in the integral (3.17) and (3.18) lead to a continuation of (3.15)–(3.18):

$$\begin{aligned} &= \int_{\mathbb{H}^n} \int_{\mathbb{H}^n} (k_2(s', x', y')\, k_1(s'', x'', y'') - k_1(s', x', y')\, k_2(s'', x'', y'')) \\ &\quad \times x''y'\, e^{i(qx'+py')}\, e^{i(qx''+py'')}\, dg'\, dg'' \\ &= \int_{\mathbb{H}^n} k_2(s', x', y')\, y'\, e^{i(qx'+py')}\, dg' \int_{\mathbb{H}^n} k_1(s'', x'', y'')\, x''\, e^{i(qx''+py'')}\, dg'' \\ &\quad - \int_{\mathbb{H}^n} k_1(s', x', y')\, y'\, e^{i(qx'+py')}\, dg' \int_{\mathbb{H}^n} k_2(s'', x'', y'')\, x''\, e^{i(qx''+py'')}\, dg'' \\ &= \frac{\partial \hat{k}_2(0, q, p)}{\partial p} \frac{\partial \hat{k}_1(0, q, p)}{\partial q} - \frac{\partial \hat{k}_1(0, q, p)}{\partial p} \frac{\partial \hat{k}_2(0, q, p)}{\partial q} \\ &= \{k_1, k_2\}. \end{aligned}$$

This finishes the proof. □

Remark 3.6. Let S, X_j, Y_j $j = 1, \ldots, n$ be vectors spanning the Lie algebra of $\mathbb{H}^n$, i.e. $[X_j, Y_j] = S$ and all other commutators vanish. Consequently the only nontrivial p-brackets among those vectors are $\{[X_j, Y_k]\} = \delta_{jk} I$. By the algebraic inheritance (Lemma 2.1) we find the only nontrivial quantum and classical brackets:

$$\frac{1}{i\hbar}[\rho_\hbar(X_j), \rho_\hbar(Y_j)] = I, \qquad \left\{\rho_{(p,q)}(X_j), \rho_{(p,q)}(Y_j)\right\} = I.$$

The role of the antiderivative $\mathcal{A}$ in (3.5) is highlighted by a comparison of (2.14) and (3.10). $\mathcal{A}$ does not only insert the multiplier $\frac{1}{i\hbar}$ in quantum commutant, it also (and this is essentially new in our construction) produces a *non-trivial classical representation* of the p-mechanical brackets.

The following corollary is very well known but we would like to incorporate it in our scheme.

Corollary 3.7. *The quantum commutator and the Poisson brackets are linear, antisymmetric, and satisfy to the Jacoby* (3.6) *and Leibniz* (3.7) *identities.*

Proof: The properties follows from the corresponding properties of p-mechanical brackets (Lemma 3.4) and conservation of algebraic identities by representations (Lemma 2.1). $\square$

As a direct consequence of the Proposition 3.5 we obtain the following statement:

Theorem 3.8. *Let a function $f(t; s, x, y)$ defined on $\mathbb{R} \times \mathbb{H}^n$ be a solution of the p-mechanical equation:*

$$\frac{d}{dt} f(t; s, x, y) = \{[f, H]\} \tag{3.19}$$

with a "Hamiltonian" $H(s, x, y)$ on $\mathbb{H}^n$. Then

(i) *The operator $f_\hbar(t; M, D) = [\rho_\hbar f](t; M, D)$ representing $f(t; s, x, y)$ under $\rho_\hbar$* (2.11) *is a solution of the Heisenberg equation*

$$\frac{d}{dt} f_\hbar(t; X, D) = \frac{1}{i\hbar}[f_\hbar, H_\hbar], \tag{3.20}$$

with the Hamiltonian operator $H_\hbar(M, D) = [\rho_\hbar H](M, D)$ from (2.11).

(ii) *The function $f_0(t; q, p) = [\rho_{(q,p)} f]$ constructed by* (2.12) *is a solution of the Hamilton equation:*

$$\frac{d}{dt} f_0(t; q, p) = \{f_0, H_0\}, \tag{3.21}$$

where the Hamiltonian function $H_0(q, p) = [\rho_{(q,p)} H]$ *is also defined by* (2.12).

Remark 3.9. We could equivalently state the universal equation (3.19) in a somewhat simpler form

$$\frac{\partial}{\partial s}\frac{d}{dt} f(t; s, x, y) = (f * H - H * f),$$

which was already proposed in Kisil (1996), but it hides the universal nature of p-mechanical bracket (3.5).

Corollary 3.10. (*Consistence of Dynamics*). *Dynamic defined by p-mechanical equation* (3.19) *and consequently by either its derivation—the Heisenberg equation* (3.20)*, or the Hamilton equation* (3.21)*—has the properties*

(i) *The identity* $C(0) = A(0) + B(0)$ *for three observables will be valid through the evolution* $C(t) = A(t) + B(t)$, $t \in \mathbb{R}_+$

(ii) *It preserves a time independent Hamiltonian.*

(iii) *Corresponding brackets* ($\{\!\![A, B]\!\!\}$, $\{A, B\}$, $[A, B]$) *of two observables A and B is again an observable evolving by the same equation.*

(iv) *The identity* $C(0) = A(0)B(0)$ *for three observables will be valid through the evolution* $C(t) = A(t)B(t)$, $t \in \mathbb{R}_+$.

(v) *The Schrödinger–Luiville and Hamilton–Heisenberg pictures of motion are equivalent.*

Proof: It is known (see Caro and Salcedo (1999)) that the above four properties are a direct consequence of those from Lemma 3.4. Again the properties are very well known for the quantum commutator and the Poisson brackets. □

Of course, it is not difficult to give a general form of a solution to the p-mechanical equation of motions:

Proposition 3.11. *Let*

$$\begin{aligned} f(t; s, x, y) &= \exp(-t\mathcal{A}H)\, f_0(s, x, y)\, \exp(t\mathcal{A}H), \\ &= \exp(-tH_{\mathcal{A}})\, f_0(s, x, y)\, \exp(tH_{\mathcal{A}}), \end{aligned} \tag{3.22}$$

be a function defined on $\mathbb{R} \times \mathbb{H}^n$. *Here in* (3.22) *H is the convolution on* $\mathbb{H}^n$ *with a Hamiltonian function* $H(s, x, y)$, $\mathcal{A}$ *is the anti-derivative operator* (3.1)*, and* $H_{\mathcal{A}}$ *is the convolution with function* $\mathcal{A}H(s, x, y)$.

Then $f(t; s, x, y)$ *from* (3.22) *satifies to the p-mechanical dynamic equation* (3.19).

Note that we never use in the above consideration any kind of limits and approximations of the type $\hbar \to 0$. Both cases of $\hbar \neq 0$ and $\hbar = 0$ were proven independently without any references to each other. On the other hand this limit does exist in the induced topology (Kirillov, 1976, § 7.3) on the dual object $\widehat{\mathbb{H}^n}$, i.e. the set of equivalence classes of unitary irreducible representation of the Heisenberg group. This topology was considered for example in Kirillov (1994, Example 7.11) and Kisil (1996) and it was shown that the set of representation $\rho_\hbar$, $\hbar \in (0, \epsilon)$ is dense in the set of representations $\rho_{(q,p)}$, $p, q \in \mathbb{R}^n$. Because we obtain both Eqs. (3.20) and (3.21) from the same source (3.19) we could conclude:

Corollary 3.12. (*The Correspondence Principle*). *Quantum dynamics is dense in classic dynamics, or in loose terms: classical dynamics a limiting case of quantum one.*

3.2. Example: The Harmonic Oscillator

We consider "the lovely pet" of quantum mechanics—the harmonic oscillator. Fortunately its consideration within p-mechanics is as well easy.

The well-known (Taylor, 1986, § 1.6) Hamiltonian of a classical harmonic oscillator is $H_0(q, p) = q^2 + p^2$ and in quantum case Hamiltonian is $H_\hbar = \hbar(M^2 + D^2)$, where operators M and D are defined in (2.7) and (2.8). It is easy to find a p-mechanical Hamiltonian that generates both quantum and classical ones.

Lemma 3.13. (i) *Let*

$$H(s, x, y) = \delta(s)\delta^{(2)}(x)\delta(y) + \delta(s)\delta(x)\delta^{(2)}(y), \tag{3.23}$$

where $\delta^{(2)}$ *is the second derivative (Kirillov and Gvishiani, 1982, § III.4.4) of the Dirac delta function* $\delta(x)$. *Then* $H_\hbar = \hbar(M^2 + D^2)$ *and* $H_0(q, p) = q^2 + p^2$ *are images of* H *under representations* $\rho_\hbar$ (2.11) *and* $\rho_{(q,p)}$ (2.12) *correspondingly.*

(ii) *The p-mechanical equation* $\dot{f} = [\![H, f]\!]$ *of the harmonic oscillator is*

$$\frac{d}{dt} f(t; s, x, y) = 2\sum_{j=1}^{n} \left(x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j} \right) f(t; s, x, y). \tag{3.24}$$

Proof: To establish first statement one verifies images of $H(s, x, y) = \delta^{(2)}(x) + \delta^{(2)}(y)$ under representations $\rho_\hbar$ (2.11) and $\rho_{(q,p)}$ (2.12) by a direct calculation. We proceed with a derivation of the Eq. (3.24). Let (Taylor, 1986, Chap. 1, (1.27))

$$X_j^r = \frac{\partial}{\partial x_j} + \frac{y_j}{2}\frac{\partial}{\partial s}, \qquad Y_j^r = \frac{\partial}{\partial y_j} - \frac{x_j}{2}\frac{\partial}{\partial s}, \tag{3.25}$$

$$X_j^l = \frac{\partial}{\partial x_j} - \frac{y_j}{2}\frac{\partial}{\partial s}, \qquad Y_j^l = \frac{\partial}{\partial y_j} + \frac{x_j}{2}\frac{\partial}{\partial s}, \quad \text{where } 1 \le j \le n, \tag{3.26}$$

be the left and the right invariant vector fields on $\mathbb{H}^n$ correspondingly. They generate the right $r(s, x, y)$ and the left $l(s, x, y)$ shifts on $L_2(\mathbb{H}^n)$ correspondingly (left invariant vector fields generate right shifts and vise verse):

$$\exp \sum_{j=1}^{n} x_j X_j^r = r(0, x, 0), \qquad \exp \sum_{j=1}^{n} x_j X_j^l = l(0, x, 0), \quad x = (x_1, \ldots, x_n)$$

$$\exp \sum_{j=1}^{n} y_j Y_j^r = r(0, y, 0), \qquad \exp \sum_{j=1}^{n} y_j Y_j^l = l(0, y, 0), \quad y = (y_1, \ldots, y_n).$$

Then we could express convolutions (2.2) with $\delta^{(2)}$ as second-order differential operators:

$$\left(\delta(s)\delta^{(2)}(x)\delta(y)\right) * f = \sum_{j=1}^{n} \left(X_j^l\right)^2 f, \qquad \left(\delta(s)\delta(x)\delta^{(2)}(y)\right) * f = \sum_{j=1}^{n} \left(Y_j^l\right)^2 f,$$

$$f * \left(\delta(s)\delta^{(2)}(x)\delta(y)\right) = \sum_{j=1}^{n} \left(X_j^r\right)^2 f, \qquad f * \left(\delta(s)\delta(x)\delta^{(2)}(y)\right) = \sum_{j=1}^{n} \left(Y_j^r\right)^2 f.$$

Therefore the commutator $[f, H]$ is

$$\begin{aligned}
[f, H] = {} & f * \left(\delta(s)\delta^{(2)}(x)\delta(y) + \delta(s)\delta(x)\delta^{(2)}(y)\right) \\
& - \left(\delta(s)\delta^{(2)}(x)\delta(y) + \delta(s)\delta(x)\delta^{(2)}(y)\right) * f \\
= {} & \sum_{j=1}^{n} \left(\left(X_j^r\right)^2 + \left(Y_j^r\right)^2 - \left(X_j^l\right)^2 - \left(Y_j^l\right)^2 \right) f \\
= {} & \sum_{j=1}^{n} \left(\left(X_j^r - X_j^l\right)\left(X_j^r + X_j^l\right) + \left(Y_j^r - Y_j^l\right)\left(Y_j^r + Y_j^l\right) \right) f \\
= {} & \sum_{j=1}^{n} \left(2y_j \frac{\partial}{\partial s}\frac{\partial}{\partial x_j} - 2x_j \frac{\partial}{\partial s}\frac{\partial}{\partial y_j} \right) f \qquad (3.27) \\
= {} & 2\frac{\partial}{\partial s} \sum_{j=1}^{n} \left(y_j \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial y_j} \right) f.
\end{aligned}$$

We substitute values from (3.25) and (3.26) in order to obtain (3.27). Finally the p-brackets (3.5) are

$$\begin{aligned}\{\![f, H]\!\} &= \mathcal{A}[f, H] \\ &= \mathcal{A}\, 2\frac{\partial}{\partial s}\sum_{j=1}^{n}\left(y_j\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial y_j}\right) f \\ &= 2\sum_{j=1}^{n}\left(y_j\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial y_j}\right) f. \end{aligned} \tag{3.28}$$

Substitution of the last formula (3.28) into p-mechanical equation (3.19) proves (3.24). □

The solution of the Eq. (3.24) is well known.

Lemma 3.14. *The evolution of an observable $f(t; s, x, y)$ of the p-mechanical harmonic oscillator is given by*

$$\begin{aligned} f(t; s, x, y) &= f_0(s, x\ \cos t + y\ \sin t, -x\ \sin t + y\ \cos t) \\ &= f_0(s, e^{-it}z), \quad \textit{for } z = x + iy, \end{aligned} \tag{3.29}$$

where $f_0(s, x, y) = f(0; s, x, y)$ is the initial value of the observable at $t = 0$.

The above evolution is transparently geometric. To preserve this property in quantum mechanics we introduce in our consideration the Segal–Bargmann (-Fock) space (Bargmann, 1961; Berezin, 1974; Berger and Coburn, 1987; Guillemin, 1984; Howe, 1980b; Segal, 1963). Let $L_2(\mathbb{C}^n, d\mu_n)$ be a space of functions on $\mathbb{C}^n$, which are square-integrable with respect to the Gaussian measure

$$d\mu_n(z) = \pi^{-n}\, e^{-z\cdot\bar{z}}\, dv(z),$$

where $dv(z) = dx\,dy$ is the Euclidean volume measure on $\mathbb{C}^n = \mathbb{R}^{2n}$. The Segal–Bargmann (Bargmann, 1961; Segal, 1963) space $F_2(\mathbb{C}^n)$ is the subspace of $L_2(\mathbb{C}^n, d\mu_n)$ consisting of all entire functions, i.e. functions $f(z)$ that satisfy

$$\frac{\partial f}{\partial \bar{z}_j} = 0, \quad 1 \le j \le n.$$

Then the Heisenberg group $\mathbb{H}^n$ acts on $F_2(\mathbb{C}^n)$ by the irreducible unitary representation

$$\beta_\hbar(s, z) f(w) = \exp(2is\hbar + i\sqrt{\hbar}zw - |z|^2)\, f(w + i\sqrt{\hbar}\bar{z}), \tag{3.30}$$

where $z = x + iy$, $(s, z) \in \mathbb{H}^n$. By the Stone–von Neumann Theorem 2.2 representations (2.6) and (3.30) are unitary equivalent.

Example 3.15. In the Segal–Bargmann representation (Berger and Coburn, 1987) creation and annihilation operators are $a_j^+ = z_j I$ and $a_j^- = \partial/\partial z_j$, respectively. The corresponding quantum Hamiltonian of harmonic oscillator is obtained by the Bargmann projection

$$T_{H(q,p)} = \frac{1}{2} P_Q \sum_{j=1}^{n} \left(q_j^2 + p_j^2\right) I = \frac{1}{2}\left(nI + \sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j}\right). \tag{3.31}$$

The right side of (3.31) is the celebrated Euler operator. It generates the well-known dynamical group (Taylor, 1986, Chap. 1, (6.35))

$$e^{itT_{H(q,p)}} f(z) = e^{int/2} f(e^{it} z), \qquad f(z) \in F_2(\mathbb{C}^n), \tag{3.32}$$

which induces rotation of the $\mathbb{C}^n$ space. Note that the frequency of the above rotation does not depend from $\hbar$.

The evolution of the classical oscillator is also given by a rotation with the same frequency, that of the phase space $\mathbb{R}^{2n}$

$$z(t) = G_t z_0 = e^{it} z_0, \qquad z(t) = p(t) + iq(t), \qquad z_0 = p_0 + iq_0. \tag{3.33}$$

The projection P_Q leads to the Segal–Bargmann representation, providing a very straightforward correspondence between quantum and classical mechanics of oscillators, in contrast to the rather complicated case of the Heisenberg representation (Taylor, 1986, Chap. 1, Prop. 7.1). The powers of z are the eigenfunctions $\phi_n(z) = z^n$ of the Hamiltonian (3.31), and the integers n are the corresponding eigenvalues. Either pure or mixed, any initial state of the oscillator remains unchanged during the (3.32) evolutions and no transitions between states are observed.

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